Name: Math 130H Lab 1

Bayes' Theorem

Definition: A partition (of a sample space of some experiment) is a collection of events that satisfy 2 conditions:

1) No outcome of the experiment belongs to more than one event (mutually exclusive)
2) Each outcome of the experiment belongs to some event (collectively exhaustive)

Think of a partition as a way of cutting up the sample space into pieces. Each piece is an event and the collection of all these pieces forms the partition. Just make sure that no 2 pieces overlap.

Ex: Suppose the experiment is spinning a roulette wheel once. Let

R = the event that the ball lands in a red slot

B = the event that the ball lands in a black slot

G = the event that the ball lands in a green slot.

Then the events R , B , and G form a partition of the sample space of this experiment because…

1) no 2 of these events overlap
2) every outcome of the experiment belongs to one of these events.

YOUR TURN: Give 2 examples of totally different experiments and partitions of their sample spaces and explain why your examples actually ARE partitions.

Step by Step Derivation of Bayes' Theorem

Suppose A_1 and A_2 form a partition of the sample space and B is some other event. Bayes' theorem is a formula that helps you calculate a conditional probability and the formula allows you to switch the 2 events in the conditional probability. That is, we will want to calculate $P(A_1|B)$ and the formula will involve $P(B|A_1)$ (i.e. the order of the 2 events in the conditional probability switched). Bayes' theorem has important, interesting, and sometimes counterintuitive results as you will see in the example at the end of this lab.

Step 1: If A_1 and A_2 form a partition of the sample space, then the 2 compound events $A_1 \cap B$ and $A_2 \cap B$ form a partition of B. That means that

1. $(A_1 \cap B) \cup (A_2 \cap B) = B$ and 2. $(A_1 \cap B) \cap (A_2 \cap B) = \emptyset$

Your turn 1: Make up your own example of an experiment and come up with a partition of its sample space that consists of 2 events. Call the 2 events in the partition A_1 and the other one A_2. Then make up another event B that overlaps (at least a little) with A_1 and with A_2. Then calculate the compound events $A_1 \cap B$ and $A_2 \cap B$ and show that they form a partition of B and explain why they form a partition of B.

Step 2: $A_1 \cap B = B \cap A_1$.
… because both of these compound events are just the list of outcomes that are in both events A_1 and B.

Step 3: $P(A_1 \cap B) = P(B \cap A_1)$
… because since both events inside the parenthesis are equal, their probabilities are equal.

Your Turn Step 4: Use the formula (the long one) for the probability of an AND for $P(A_1 \cap B)$, do the same for $P(B \cap A_1)$, set the 2 answers equal to each other and solve for $P(A_1|B)$. (Hint: Your answer should be a fraction with $P(B)$ in the denominator)

Step 5: Working on the denominator from the your answer above, since from step 1
$(A_1 \cap B) \cup (A_2 \cap B) = B$, we get $P(B) = P((A_1 \cap B) \cup (A_2 \cap B))$.

Your Turn 5: Use the probability of an OR formula, then the probability of an AND formula to find $P((A_1 \cap B) \cup (A_2 \cap B))$.
(Hints:

1. The 2 compound events in the parenthesis are disjoint from step 1.
2. Which probability of an OR formula are you going to use?
3. Your answer should have 2 terms in it, each term should have 2 probabilities multiplied together, and there should be no OR or AND symbols in the final answer.

Your Turn Step 6: Substitute your answer from step 5 for the denominator in step 4 to obtain Bayes' Theorem. Write the formula below.

Your Turn: Bayes' theorem can be extended to a partition of the sample space with more events than just 2. If A_1, A_2, and A_3 form a partition of the sample space and B is some other event, extend the formula you derived above for this situation.

Your Turn Problem (book section 5.7, #34 modified): **Elisa Test** The standard test for the HIV virus is the Elisa test, which test for the presence of HIV antibodies. If an individual does not have the HIV virus, the test will come back negative for the presence of HIV antibodies 99.8% of the time and will come back positive for the presence of HIV antibodies 0.2% of the time (a false positive, meaning that even though the individual doesn't have HIV, the test will say they do and be wrong 0.2% of the time. No test is perfect, but if the test is going to make this kind of mistake, the chance of it making such a mistake better be small!). If an individual has the HIV virus, the test will come back positive 99.7% of the time and will come back negative 0.3% of the time (a false negative, meaning that even though the patient has HIV, the test will make a mistake and say that the patient doesn't have HIV). Approximately 0.7% of the world population has the HIV virus. Let

A_1 = The event that a person has HIV

A_2 = The event that a person does not have HIV

B = The event that a person's Elisa test result is positive for the presence of HIV antibodies

a) Find

$P(A_1) =$ $\qquad$ $P(B|A_1) =$ $\qquad$ $P(A_2) =$ $\qquad$ $P(B|A_2) =$

(Hint: Most of these are given in the problem and the 3rd one is super easy to figure out)

b) Find the probability that a person has the HIV virus given that they tested positive for the presence of HIV antibodies from the Elisa test. Do this in 3 steps…

1) Write the notation for the probability that you are being asked to find
2) Write down the formula for Bayes' theorem for a 2 event partition
3) Plug in the numbers and calculate

Reflection: Look at your answer from the last problem. You just found the probability that a person actually has HIV given that they tested positive for HIV. The result is really interesting because we expect the answer to be high, but it isn't too high. If you calculate 100% - the probability you found (what is that number?), this is the probability that a person tests positive for HIV but actually doesn't have it. We want this number to be super-small, but it isn't! In other words, if you test positive for an illness (with really good machines) you still need to get a second opinion because there is a real chance that the test may have given you an incorrect prediction.